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Global Solutions of the Schrödinger Equation with Coulomb plus Linear Potential

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Normalizable solutions of the Schrödinger equation with a potential of the type used to describe the quark–antiquark or quark–quark interactions are discussed. The eigenenergies are computed by connecting the series solutions near the origin with the formal asymptotic solutions. The Regge trajectories for the charm–anticharm and the bottom–antibottom systems are obtained in that way. © 1994 Academic Press, Inc.

1. INTRODUCTION

Almost two decades of meson and baryon spectroscopy have contributed to reinforce the nonrelativistic quark model of hadrons. There are recent reviews of the topic [1, 2] where references to the original papers can be found. Mesons and baryons are successfully described as bound states of quarks and antiquarks interacting via a potential and obeying the nonrelativistic Schrödinger equation with Breit–Fermi corrections.

Several phenomenological potentials have been used to fit the experimental masses of the hadrons. However, the "Cornell" or "Coulomb plus linear" potential,

$$V(r) = -a'/r + b + cr, \qquad (1.1)$$

seems to be the most popular, due to the fact that its short- and long-range behaviours are inspired respectively by perturbative and lattice quantum chromodynamics [3], although the values of the parameters are adjusted phenomenologically to give the best fit.

Analytical solutions of the Schrödinger equation with potential (1.1) valid in the whole range of r from 0 to ∞ have not yet been obtained. Two independent solutions, in the form of series expansions in powers of r, can be immediately found. Also, two other independent solutions, expressed as asymptotic expansions useful for large values of r, can be easily obtained. The problem, however, is to connect the solutions near the origin with the asymptotic ones, in order to get global (i.e., valid for all r from 0 to ∞) solutions. Specifically, the determination of the masses of the mesons requires the obtention of eigenvalues of the Schrödinger equation for which global solutions can be found that are well behaved (i.e., regular at 0 and ∞). A lot of numerical methods have been proposed to find those eigenvalues. References can be found in recent contributions [4–7]. But, although very different schemes have been used, all of them lie on either the numerical integration of the differential equation or the replacement of the potential by an approximated one that allows to use analytical methods.

In this paper we apply a different approach, based on connection relations among the solutions near the origin and the asymptotic ones, to the determination of eigenvalues. The connection problem for second-order linear differential equations with two irregular singular points was considered by Naundorf [8], who succeeded in finding, under certain conditions, connection factors allowing us to express the power series solutions as combinations of the asymptotic expansions. Those factors obey a system of linear equations whose coefficients are obtained by (numerical, in general) summation of series. Naundorf's method is applicable, and becomes considerably simplified, in the case of one of the singular points being regular, as it happens for the Schrödinger equation with potential (1.1).

We recall in Section 2 the solution of Naundorf to the connection problem and particularize it to the Schrödinger equation with Coulomb plus linear potential. In Section 3 we examine the conditions to be satisfied by the connection factors in order to have physically well-behaved solutions. Section 4 shows some energy eigenvalues obtained for the sets of parameters of the potential used to describe respectively the charmonium and the bottomonium states. Finally, in Section 5, we add some comments.

2. THE CONNECTION PROBLEM

The reduced radial Schrödinger equation for a particle of mass m, energy E, and angular momentum l, in a potential $V(r)$, reads

$$\left(\frac{-\hbar^2}{2m}\left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right) + V(r) - E\right) u(r) = 0, \qquad (2.1)$$

that, in the case of the potential being given by Eq. (1.1), turns out

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$$\frac{d^2}{dr^2}u(r) + \left(\frac{2m}{\hbar^2}\left(-cr + E - b + \frac{a'}{r}\right) - \frac{l(l+1)}{r^2}\right)u(r) = 0. \tag{2.2}$$

Let us introduce, as usual, dimensionless variable

$$z \equiv (2mc/\hbar^2)^{1/3}\, r \tag{2.3}$$

and parameters

$$\varepsilon \equiv (2m/\hbar^2c^2)^{1/3}(E - b), \quad a \equiv (4m^2/\hbar^4 c)^{1/3}\, a'. \tag{2.4}$$

Let us also denote by $v(z)$ the function $u(r)$ in terms of the new variable. Equation (2.2) then becomes

$$\frac{d^2}{dz^2}v(z) + \left(-z + \varepsilon + \frac{a}{z} - \frac{l(l+1)}{z^2}\right)v(z) = 0. \tag{2.5}$$

Certain particular values of the parameters lead to well-known differential equations. For instance, for $a = 0$ and $l = 0$ one obtains the equation obeyed by the Airy functions of variable $z - \varepsilon$. Also, for $\varepsilon = 0$ and $a = 0$ the function $z^{-1/2}v(z)$ obeys a Bessel equation of order $(2l + 1)/3$ and variable $(2/3)e^{i\pi/2}z^{3/2}$. Here we are interested, however, in the general case, when all terms in (2.5) are assumed to be present.

Although Eq. (2.5) has already a dimensionless form, its asymptotic solutions contain fractionary powers of the variable. To avoid this, it is preferable to introduce a new variable

$$t \equiv z^{1/2} \tag{2.6}$$

in terms of which, and replacing $v(z)$ by $w(t)$, Eq. (2.5) becomes

$$t^2\frac{d^2w(t)}{dt^2} - t\frac{dw(t)}{dt} + (-4t^6 + 4\varepsilon t^4 + 4at^2 - 4l(l+1))w(t) = 0. \tag{2.7}$$

This form of the Schrödinger equation is suitable for application of Naundorf's method.

Equation (2.7) presents a regular singular point at $t = 0$ and an irregular (of rank 3) singular one at $t = \infty$. Power series solutions of the form

$$w(t) = \sum_{n=0}^{\infty} c_n t^{n+\rho}, \quad c_0 = 1, \quad 0 < |t| < \infty, \tag{2.8}$$

can be found for two different values of the exponent ρ, namely,

$$\rho_1 = 2l + 2, \quad \rho_2 = -2l, \tag{2.9}$$

and coefficients c_n obeying the recurrence relation

$$((n+\rho)(n+\rho-2) - 4l(l+1))\, c_n + 4ac_{n-2} + 4\varepsilon c_{n-4} - 4c_{n-6} = 0. \tag{2.10}$$

For l complex or real different from an integer or half integer that recurrence does not present any difficulty; therefore, two independent solutions of the form (2.8) are found. For integer or half integer l, instead, only one of the solutions, that corresponding to positive ρ, is of the form (2.8); the other independent solution contains logarithmic terms. Of course, the case of nonnegative integer l is the most relevant from the physical point of view. Nevertheless, since we are interested in solutions that are well behaved at the singular points, we can disregard those presenting such logarithmic terms and limit our subsequent discussion to solutions of the form (2.8).

Two independent formal solutions, useful for large t, can be found by the substitution of

$$w_{asy}^{(k)} = \exp\left(\sum_{j=1}^{3}\frac{\alpha_j^{(k)}}{j}t^j\right) t^{\mu^{(k)}} \sum_{s=0}^{\infty} h_s^{(k)} t^{-s}, \qquad h_0^{(k)} = 1,\ k = 1, 2, \tag{2.11}$$

for $w(t)$ in Eq. (2.7). The values of the exponents are

$$\alpha_3^{(1)} = -2, \quad \alpha_3^{(2)} = 2, \tag{2.12a}$$

$$\alpha_2^{(1)} = 0, \quad \alpha_2^{(2)} = 0, \tag{2.12b}$$

$$\alpha_1^{(1)} = \varepsilon, \quad \alpha_1^{(2)} = -\varepsilon, \tag{2.12c}$$

$$\mu^{(1)} = -\tfrac{1}{2}, \quad \mu^{(2)} = -\tfrac{1}{2}, \tag{2.12d}$$

and the recurrence for the coefficients $h_s^{(k)}$ becomes

$$\begin{aligned} 2s\alpha_3 h_s = {} & (\alpha_1^2 + 4a)h_{s-1} + (2\mu - 1 - 2(s-2))\alpha_1 h_{s-2} \\ & + (\mu(\mu - 2) - 4l(l+1) \\ & + (s-3)(s-1-2\mu))h_{s-3}, \end{aligned} \tag{2.13}$$

the superindex (k) being understood. Bearing in mind that

$$\alpha_3^{(2)} = -\alpha_3^{(1)}, \quad \alpha_1^{(2)} = -\alpha_1^{(1)} \tag{2.14}$$

and having taken

$$h_0^{(2)} = h_0^{(1)} = 1, \tag{2.15}$$

it is immediate to check that

$$h_s^{(2)} = (-1)^s h_s^{(1)}. \tag{2.16}$$

This leads to a symmetry,

$$w_{asy}^{(1)}(e^{\pm i\pi}t) = e^{\mp i\pi/2} w_{asy}^{(2)}(t), \tag{2.17}$$

between the formal solutions, that obey also the circuital relation

$$w_{asy}^{(k)}(e^{2i\pi}t) = e^{-i\pi} w_{asy}^{(k)}(t), \quad k = 1, 2. \tag{2.18}$$

The connection problem consists in obtaining, for a given ray (fixed $\arg(t)$), numbers T_k, allowing us to express the asymptotic behaviour of the power series solution as a linear combination of the two formal solutions, i.e.,

$$w(t) \sim \sum_{k=1}^{2} T_k w_{asy}^{(k)}(t), \quad |t| \to \infty, \tag{2.19}$$

the rigorous meaning of symbol $\sim$ being explained in Ref. [8]. In what follows we report the solution to that problem given by Naundorf, particularized to Eq. (2.7), omitting all proofs and details of lemmas and theorems, that the interested reader can find in Ref. [8]. Moreover, the obtention of connection factors requires us to consider different regions of the complex t-plane, namely, the sectors

$$S_{k,p} = \{t; |\arg(\alpha_3^{(k)} t^3) - 2p\pi| < \pi\}, \qquad k = 1, 2, \ p \text{ an integer.} \tag{2.20}$$

All sectors have central angle $2\pi/3$, $S_{k,p}$ and $S_{k,p+1}$ being separated by the ray with argument $((2p + 1)\pi - \arg(\alpha_3^{(k)}))/3$. As we are interested mainly in the physical problem, we will look for connection factors T_k on the positive real axis of the t-plane, that is, for $\arg(t) = 0$.

The Naundorf's method makes use of the Heaviside's exponential series

$$\sum_{n=-\infty}^{\infty} \frac{x^{n+\delta}}{(n + \delta)!}$$

that is equal to $\exp(x)$ for every integer-valued δ and, although divergent everywhere in the complex x-plane, that verifies

$$\exp(x) \sim \sum_{n=-\infty}^{\infty} \frac{x^{n+\delta}}{(n + \delta)!}, \quad |\arg(x)| < \pi, \tag{2.21}$$

for any other value of δ. Accordingly, the factor $\exp(\alpha_3^{(k)} t^3/3)$ in $w_{asy}^{(k)}(t)$ admits three linearly independent series expansions, indexed by $L = 0, 1, 2$,

$$\exp\left(\frac{\alpha_3^{(k)}}{3} t^3\right) \sim \sum_{n=-\infty}^{\infty} \frac{((\alpha_3^{(k)}/3)t^3)^{n+(\delta+L)/3}}{(n + (\delta + L)/3)!}, \quad |\arg(\alpha_3^{(k)} t^3)| < \pi. \tag{2.22}$$

Multiplication of this expression by the Taylor expansion of $\exp(\alpha_1^{(k)} t)$ gives for the exponential in the right-hand side of Eq. (2.11),

$$\exp\left(\sum_{j=1}^{3} \frac{\alpha_j^{(k)}}{j} t^j\right) \sim \sum_{n=-\infty}^{\infty} g_n^{(k,L)} t^{n+\delta}, \quad t \in S_{k,0}, \tag{2.23}$$

with

$$g_n^{(k,L)} = \sum_{q=-\infty}^{[(n-L)/3]} \frac{(\alpha_3^{(k)}/3)^{q+(\delta+L)/3}}{(q + (\delta + L)/3)!} \frac{(\alpha_1^{(k)})^{n-L-3q}}{(n - L - 3q)!}, \qquad k = 1, 2, \quad L = 0, 1, 2, \tag{2.24}$$

where the symbol [ ] stands for the integer part. In other sectors one would obtain

$$\exp\left(\sum_{j=1}^{3} \frac{\alpha_j^{(k)}}{j} t^j\right) \sim \exp\left(-2i\pi p \frac{L + \delta}{3}\right) \sum_{n=-\infty}^{\infty} g_n^{(k,L)} t^{n+\delta}, \quad t \in S_{k,p}. \tag{2.25}$$

Equation (2.23) allows one to write Eq. (2.11) in the form

$$w_{asy}^{(k)} \sim \sum_{n=-\infty}^{\infty} f_n^{(k,L)} t^{n+\rho}, \quad k = 1, 2, \quad L = 0, 1, 2, \quad t \in S_{k,0}, \tag{2.26}$$

with

$$f_n^{(k,L)} = \sum_{s=0}^{\infty} g_{n+s}^{(k,L)} h_s^{(k)} \tag{2.27}$$

and the value of δ being chosen so as to satisfy

$$\mu^{(k)} + \delta = \rho. \tag{2.28}$$

As proved in Ref. [8], comparison of Eqs. (2.8) and (2.26) allows one to conclude that the coefficients c_n of the convergent series solution depend linearly on the $f_n^{(k,L)}$,

$$c_n = \sum_{k=1}^{2} \sum_{L=0}^{2} \beta_{k,L} f_n^{(k,L)}, \tag{2.29}$$

for all n larger than a certain N in such a way that

$$4(1 + |\varepsilon| + |a| + |l(l + 1)|) < |n + 6 + \rho|(|n + 5 + \rho| - 1). \tag{2.30}$$

Once the (complex) constants $\beta_{k,L}$ are obtained from the system of Eqs. (2.29) with $n = N + 1, N + 2, \ldots, N + 6$, one can define the coefficients

$$T_{k,p} \equiv \sum_{L=0}^{2} \exp(2i\pi p(L + \rho - \mu_k)/3)\beta_{k,L}, \quad p \text{ an integer,} \tag{2.31}$$

in terms of which the connection factors become

$$T_k = T_{k,p} \quad \text{if } t \in S_{k,p}, \tag{2.32a}$$

$$T_k = \tfrac{1}{2}(T_{k,p} + T_{k,p+1}) \tag{2.32b}$$

if t is on the boundary ray that separates $S_{k,p}$ and $S_{k,p+1}$.

3. NORMALIZABLE SOLUTIONS

As stated in the Introduction, we are interested in solutions to the physical problem of eigenenergies; that is, we are looking for real values of the parameter ε for which solutions regular at the singular points do exist, the angular momentum parameter l being assumed to be nonnegative real. The solution physically acceptable at $t = 0$ is of the form (2.8) with $\rho = 2l + 2$. On the other hand, as $t \to \infty$, the formal solution $w_{as}^{(1)}(t)$ is well behaved over the ray $\arg(t) = 0$, whereas $w_{as}^{(2)}(t)$ diverges. Obviously, the energy parameter ε must be such that the connection factor T_2 in Eq. (2.19) vanishes over the ray $\arg(t) = 0$, or, equivalently, in view of Eq. (2.32a),

$$T_{2,0} = 0, \tag{3.1}$$

that, bearing in mind Eq. (2.31), can be written

$$\sum_{L=0}^{2} \beta_{2,L} = 0. \tag{3.2}$$

Up to now we have followed the notation of Ref. [8]. In order to benefit from the symmetries of the particular case under consideration, it is convenient to write the linear system of Eqs. (2.29) in terms of new unknowns,

$$\hat{\beta}_{k,L} \equiv \left(\frac{\alpha_3^{(k)}}{3}\right)^{(\delta+L)/3} \beta_{k,L} \tag{3.3}$$

and coefficients

$$\hat{f}_n^{(k,L)} \equiv \left(\frac{\alpha_3^{(k)}}{3}\right)^{-(\delta+L)/3} f_n^{(k,L)} = \sum_{s=0}^{\infty} \hat{g}_{n+s}^{(k,L)} h_s^{(k)}, \tag{3.4}$$

with

$$\hat{g}_n^{(k,L)} \equiv \left(\frac{\alpha_3^{(k)}}{3}\right)^{-(\delta+L)/3} g_n^{(k,L)} = \sum_{q=-\infty}^{[(n-L)/3]} \frac{(\alpha_3^{(k)}/3)^q}{(q + (\delta+L)/3)!} \frac{(\alpha_1^{(k)})^{n-L-3q}}{(n-L-3q)!}. \tag{3.5}$$

Obviously, the new system reads

$$\sum_{k=1}^{2}\sum_{L=0}^{2} \hat{f}_n^{(k,L)} \hat{\beta}_{k,L} = c_n, \quad n = N+1, N+2, \ldots, N+6. \tag{3.6}$$

The advantage of the new notation lies on the fact that, in view of Eqs. (2.14),

$$\hat{g}_n^{(2,L)} = (-1)^{n-L} \hat{g}_n^{(1,L)} \tag{3.7}$$

and, bearing in mind Eq. (2.16),

$$\hat{f}_n^{(2,L)} = (-1)^{n-L} \hat{f}_n^{(1,L)}. \tag{3.8}$$

This symmetry property allows one to decouple the system (3.6) into two systems of three equations for a new set of unknowns,

$$\gamma_0 \equiv \hat{\beta}_{2,0} + \hat{\beta}_{1,0}, \quad \gamma_1 \equiv \hat{\beta}_{2,1} - \hat{\beta}_{1,1}, \quad \gamma_2 \equiv \hat{\beta}_{2,2} + \hat{\beta}_{1,2}, \tag{3.9a}$$

$$\eta_0 \equiv \hat{\beta}_{2,0} - \hat{\beta}_{1,0}, \quad \eta_1 \equiv \hat{\beta}_{2,1} + \hat{\beta}_{1,1}, \quad \eta_2 \equiv \hat{\beta}_{2,2} - \hat{\beta}_{1,2}. \tag{3.9b}$$

The new systems of equations are

$$\sum_{L=0} \hat{f}_n^{(2,L)} \gamma_L = c_n, \quad n = n_0, n_0+2, n_0+4, \tag{3.10a}$$

$$\sum_{L=0} \hat{f}_n^{(2,L)} \eta_L = c_n, \quad n = n_0+1, n_0+3, n_0+5, \tag{3.10b}$$

n_0 being any even integer larger than N. It is obvious from Eq. (2.10) that the c_n are equal to 0 for odd n. The only solution of system (3.10b) is, therefore,

$$\eta_L = 0, \quad L = 0, 1, 2. \tag{3.11}$$

This implies that

$$\gamma_L = 2\hat{\beta}_{2,L}, \quad L = 0, 1, 2, \tag{3.12}$$

and, consequently, the condition (3.2) for having physically acceptable solutions becomes

$$\sum_{L=0}^{2} (2/3)^{-L/3} \gamma_L = 0. \tag{3.13}$$

Both this equation and the fact that the γ's are the solutions of (3.10a) lead to a new form of the condition for the existence of normalizable solutions,

$$\det \begin{vmatrix} \hat f^{(2,0)}_{n_0} & \hat f^{(2,1)}_{n_0} & \hat f^{(2,2)}_{n_0} & c_{n_0} \\ \hat f^{(2,0)}_{n_0+2} & \hat f^{(2,1)}_{n_0+2} & \hat f^{(2,2)}_{n_0+2} & c_{n_0+2} \\ \hat f^{(2,0)}_{n_0+4} & \hat f^{(2,1)}_{n_0+4} & \hat f^{(2,2)}_{n_0+4} & c_{n_0+4} \\ 1 & (3/2)^{1/3} & (3/2)^{2/3} & 0 \end{vmatrix} = 0, \quad (3.14)$$

the dependence on the energy parameter ε being implicit in the coefficients $\hat f^{(2,L)}_n$ and c_n.

4. CHARMONIUM AND BOTTOMONIUM STATES

We have tested the feasibility of the above-described method by applying it to the Schrödinger equation with a Coulomb plus linear potential, Eq. (1.1), that reproduces the measured masses of the known charmonium and bottomonium states. The parameters take the values [2]

$$a' = \tfrac{4}{3}\cdot 0.3548, \quad b = -0.5466\,\text{GeV}, \quad c = 0.2079\,(\text{GeV})^2, \quad (4.1)$$

the radial distance r being expressed in $(\text{GeV})^{-1}$. Assuming for the masses of the charm and bottom quarks, respectively [2],

$$m_c = 1.632\,\text{GeV}, \quad m_b = 5.015\,\text{GeV}, \quad (4.2)$$

the parameter a in Eq. (2.5) takes the values

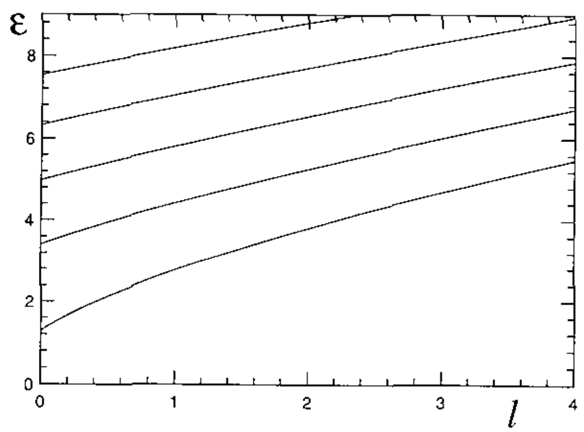

FIG. 1. Energies of the charmonium for continuously varying angular momentum. Only the five lowest values of the dimensionless energy parameter ε are shown. The corresponding values of the energy E are obtained from those of ε by means of Eq. (2.4) of the text, with the potential parameters and mass given by Eqs. (4.1) to (4.3).

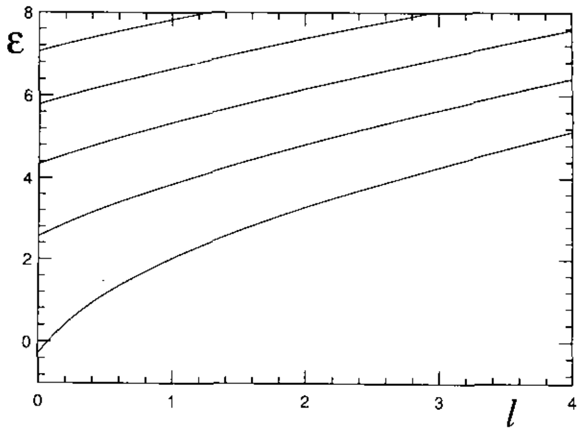

FIG. 2. Energies of the bottomonium for continuously varying angular momentum. The comments in caption of Fig. 1 are applicable also here.

$$a_c = 1.1069, \quad a_b = 2.3395. \quad (4.3)$$

By using Eq. (3.14), we have obtained some eigenvalues of the energy parameter ε in both cases of charmonium and bottomonium. Figures 1 and 2 show how those eigenvalues vary with the angular momentum, i.e., the Regge trajectories.

A few comments about the computational procedure are in order. There is no difficulty at all for the computation of the coefficients c_n. The $\hat f^{(k,L)}_n$, however, are obtained by summation of the series in Eq. (2.27) that, although convergent [8] as fast as the geometric series

$$\sum_{s=0}^{\infty} |\alpha_3^{(k)}/(\alpha_3^{(2)} - \alpha_3^{(1)})|^{s/3},$$

they may require the inclusion of many terms if one desires a reasonable accuracy in the determination of the eigenenergies. The number of needed terms in the series increases as ε grows. A test of the reliability of the algorithm is provided by the fact that the solutions of Eq. (3.14) for $a = 0$ and $l = 0$ are the absolute values of the zeros of the Airy function Ai. More terms in the series require also more precision in their computation. For ε about 10, around 400 terms calculated with quadruple precision are needed to determine the eigenvalues with at least eight correct digits. For larger values of ε the method should not be recommendable.

The computation of h_s encounters overflow problems when large values of s need to be taken into account in Eq. (3.4). For that reason it is preferable to organize the evaluation of the $\hat f^{(k,L)}_n$ according to a more suitable form of Eq. (3.4),

$$\hat f_n = \hat g_n h_0 \left(1 + \frac{\hat g_{n+1}}{\hat g_n}\frac{h_1}{h_0}\left(1 + \frac{\hat g_{n+2}}{\hat g_{n+1}}\frac{h_2}{h_2}(1 + \ldots)\right)\right), \quad (4.4)$$

the superindices k and L being omitted here and in what follows. The quotients

$$\theta_s \equiv \frac{h_s}{h_{s-1}} \tag{4.5}$$

can be easily obtained by means of the recurrence

$$\begin{aligned}
2\alpha_3\theta_1 &= \alpha_1^2 + 4a,\\
4\alpha_3\theta_2 &= \alpha_1^2 + 4a + \theta_1^{-1}(2\mu - 1)\alpha_1,\\
6\alpha_3\theta_3 &= \alpha_1^2 + 4a + \theta_2^{-1}((2\mu - 1 - 2)\alpha_1 + \theta_1^{-1}\\
&\qquad (\mu(\mu - 2) - 4l(l+1))),\\
2s\alpha_3\theta_s &= \alpha_1^2 + 4a + \theta_{s-1}^{-1}((2\mu - 1 - 2(s-2))\alpha_1\\
&\qquad + \theta_{s-2}^{-1}(\mu(\mu - 2) - 4l(l+1) + (s-3)(s-1-2\mu))).
\end{aligned} \tag{4.6}$$

For the quotients

$$\frac{\hat{g}_{n+s}}{\hat{g}_{n+s-1}}$$

no special algorithm is needed; use of Eq. (3.5), followed by obvious simplifications, is adequate.

5. FINAL COMMENTS

The Naundorf's method, adapted to the problem under consideration, reveals it to be adequate to determine the eigenenergies of the Cornell potential. As we have seen, they are obtained as zeros of the determinant of a 4 × 4 matrix. The procedure is not fully analytic, since 12 elements of the matrix are obtained numerically, but it presents the advantage of avoiding the numerical integration of the differential equation.

The method is equally useful to obtain the values of the parameters in the potential for which a particular eigenenergy would be obtained. The equation giving those values is, of course, Eq. (3.14) with the particular value of ε, a being now the unknown.

Equation (3.14) becomes considerably simplified if one is interested in obtaining the critical values of the coupling constant, i.e., those values of a for which a zero eigenenergy exists. This problem, traditionally considered cumbersome for conventional potentials [9], does not present any special difficulty in the case of confining ones. For $\varepsilon = 0$ the expression (3.5) for $\hat{g}_n^{(k,L)}$ turns much simpler,

$$\hat{g}_n^{(k,L)} = \begin{cases} \dfrac{\left(\dfrac{\alpha_3^{(k)}}{3}\right)^{(n-L)/3}}{\left(\dfrac{n+\delta}{3}\right)!}, & \text{for } n - L = 0 \bmod 3,\\[2ex] 0, & \text{for } n - L \neq 0 \bmod 3; \end{cases} \tag{5.1}$$

and the algorithm (4.4) to obtain the $\hat{f}_n^{(k,L)}$ becomes

$$\hat{f}_n = \frac{\left(\dfrac{\alpha_3}{3}\right)^{(n-L+\Delta)/3}}{((n+\delta+\Delta)/3)!} h_\Delta \left(1 + \frac{\alpha_3}{n+\delta+3+\Delta}\frac{h_{3+\Delta}}{h_\Delta} \times \left(1 + \frac{\alpha_3}{n+\delta+6+\Delta}\frac{h_{6+\Delta}}{h_{3+\Delta}}(1 + \ldots)\right)\right), \tag{5.2}$$

Δ being the lowest nonegative integer such that

$$n - L + \Delta = 0 \bmod 3. \tag{5.3}$$

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